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Liquidity, Complexity and Scale in Private Markets

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The market for private assets is distinct from public markets along many dimensions. Although the underlying economic exposures of private and public investments may be quite similar, private assets are generally far less liquid compared with assets that trade on public exchanges or over-the-counter (OTC) markets. But while holding illiquid assets means the investor must be willing to forgo the ability to dispose of the asset over some horizon, illiquidity can materially affect expected returns. The additional return compensation that patient investors can expect to earn from holding such assets is the so-called illiquidity premium.

Hibbert et al. (2009) define the illiquidity premium as follows:

> *The liquidity premium[1] for a given security can be thought of as being the price discount or excess return/yield offered by the security relative to some hypothetical, perfectly liquid security with otherwise equivalent characteristics.*

Assets may be illiquid for a number of reasons. A primary determinant of illiquidity is transaction costs, which can range from a few basis points for large cap equities to hundreds of basis points for private assets. In fact, a market for private assets may not exist at all in some periods at any price. Additionally, the complexity of the underlying asset or investment structure can lead to illiquidity by limiting the set of potential investors to those with the ability to analyze such opportunities. This is true in both public and private markets, but particularly so in the private arena, where assets have a wide array of characteristics that can make them difficult to understand. This speaks to the importance of scale in the market for private investments: Private assets can often be complex in nature, leading to a dearth of liquidity by limiting the natural set of potential buyers to those with sufficient scale and resources to pursue such opportunities. Hence, while patience is necessary to earn the liquidity risk premium, scale is necessary to earn the complexity risk premium.

MODELS OF THE ILLIQUIDITY PREMIUM

The primary justification for the existence of the illiquidity premium is transaction costs. Of course, transaction costs can embody a wide range of market frictions. Factors such as bid-ask spreads, quote depth, market impact and brokerage fees affect trading costs, but costs also stem from less obvious sources, such as the search costs and uncertainty associated with locating a willing future buyer (Duffie et al. 2007). Additionally, although investors may have a reasonable assessment of average, or expected, future transaction costs, there is the

1 PIMCO prefers to use the term illiquidity premium.

potential for liquidity to dry up in the future (Ang et al. 2014). Ultimately, all of these factors affect the distribution of future trading costs and hence the illiquidity premium.

Below, we review four models from the academic literature on the illiquidity premium. Though by no means exhaustive, these papers have made meaningful contributions to investors' understanding of the sources and magnitude of the illiquidity premium.

Amihud and Mendelson (1986)

Amihud and Mendelson use a model to show how the illiquidity premium is influenced by the existence of short-term and long-term investors. Long-term investors are characterized by relative patience compared with their more myopic counterparts and thus have less need for liquidity. As such, long-term investors are able to amortize trading costs over a longer horizon, meaning that the cost per unit of time paid by patient investors is lower than that paid by those with more immediate liquidity needs. The main results of the paper are that 1) patient investors can earn an illiquidity premium by specializing in less liquid segments of the market and 2) the relationship between the expected return and the holding horizon is concave, meaning that the return premium increases at a diminishing rate with the time horizon.

The authors start by assuming that investors are risk neutral. There is a series of assets that are identical except for their trading costs. Each asset i pays a fixed dividend d_i and is characterized by a sell cost of C_i (for simplicity, the model assumes costs are only incurred at the time of sale). The authors show that if all investors are characterized by the same trading intensity μ (which measures the probability that the investor will need to exit the market in a given period), the expected return on all assets is given by

$$E(r_i) = r^f + \mu \frac{C_i}{P^i}. \tag{1}$$

Equation 1 says that in the absence of any liquidity needs (μ = 0), the expected return on an asset is equal to the risk-free rate (recall that investors are risk neutral in the model). However, as investors have some nonzero probability of needing liquidity, they command lower prices for the asset (and hence higher returns) as liquidity needs and transaction costs increase. But all assets earn the same risk-free return after costs, and the only factor affecting relative prices is trading costs.

The more interesting result occurs when the authors allow for a range of liquidity needs, characterized by investors with different levels of trading intensity, μ. In this case, the model of Amihud and Mendelson results in a set of clientele effects, whereby different investor types specialize by holding assets with varying degrees of transaction costs. Specifically, investors with the greatest liquidity needs hold the most liquid asset and earn the risk-free rate (after transaction costs). More-patient investors hold higher-cost investments and, in doing so, are able to command a liquidity risk premium relative to their less patient peers. Specifically, the authors show that if type-j investors hold security i, then security i has an expected return equal to

$$E(r_i) = r^f + \left(r^{*j} - r^f\right) + \mu^j \frac{C_i}{P^i}. \tag{2}$$

Equation 2 shows that when investors have heterogeneous liquidity needs, patient investors are able to command an illiquidity premium equal to ($r^{*j} - r^f$), where r^{*j} is the type j investor's required liquidity- adjusted return. This parameter is a function of the relative trading costs and the trading intensities of other market participants. In a subsequent study comparing liquid and illiquid publicly traded U.S. equities, Amihud (2002) finds a liquidity risk premium of about 1.3% per year.

For intuition, we describe a simple stylized example of the model. Assume that the bid-ask spread is 1% on a liquid security and 3% on an illiquid security, and that the risk-free rate is 1.5%. The market comprises two investor types: a short-term investor whose average holding horizon is six months and a long-term investor with a horizon of five years. If investors are risk neutral, then the liquid security's expected return net of transaction costs is the same as the risk-free rate. Hence, the expected gross return on the liquid asset is 1.5% + (1%*2) = 3.5%, because the short-term investor makes two round trips a year. Of course, the long-term investor can also buy the liquid security at a yield of 3.5%, so the expected net return to the long-term investor from holding the same asset is 3.5% - (1%/5) = 3.3%, which is 1.8% higher than what the myopic investor earns after transaction costs. At a minimum, therefore, the illiquid security needs to yield 3.3% net to the long-term investor, who otherwise would universally prefer to buy the liquid security. Thus, in equilibrium the illiquid security's gross yield from the perspective of the long-term investor is the expected net yield on the liquid security plus compensation for transaction costs – that is, 3.3% + (3%/5) = 3.9%. The long-term investor therefore holds the illiquid asset and the short-term investor the liquid asset, and each earns net returns of 1.5% and 3.3%, respectively. This difference is the so-called liquidity risk premium. It is formally equal to the difference in trading frequencies times the cost of trading the liquid security:

$$LP = (f_{ST} - f_{LT})C_{liq} \quad (3)$$

where f_{ST} and f_{LT} are the trading frequencies of short-term and long-term investors, respectively, and C_{liq} is the trading cost of the liquid security. Hence, a liquidity risk premium is (2-0.2)*1% = 1.8%. So the longer the patient investor's horizon, the higher the premium and the shorter the short-term investor's horizon, the higher the premium. *Interestingly, the liquidity risk premium moves with the bid-ask on the liquid security and does not depend on the bid-ask of the illiquid security.*

Acharya and Pedersen (2005)

One can think of Amihud and Mendelson's model as representing how average liquidity is priced in equilibrium. More-recent work by Acharya and Pedersen speaks to the covariance risk of assets with respect to marketwide liquidity. In their paper, the authors quote two sources that provide an anecdotal justification for covariance risk as a priced factor:

> *The possibility that liquidity might disappear from a market, and so not be available when it is needed, is a big source of risk to an investor.*
>
> *The Economist,* 23 September, 1999

> *... there is also broad belief among users of financial liquidity – traders, investors, and central bankers – that the principal challenge is not the average level of financial liquidity ... but its variability and uncertainty ...*
>
> "Avinash D. Persaud, Liquidity Black Holes,"
> *Risk Magazine,* 2003

These statements highlight the fact that investors should command an illiquidity premium not only for the average, or expected, level of transaction costs but also the sensitivity to marketwide liquidity. Indeed, several academic studies have confirmed the fact that liquid markets can become suddenly and unexpectedly illiquid. For example, Krishnamurthy et al. (2014) document the existence of buyer strikes in money markets, and Borio (2004) finds similar effects with respect to liquidity during several global crises. Assets whose illiquidity covaries positively with aggregate illiquidity should command higher premiums as compensation for liquidity drying up when it is most needed. The main result of the model is a liquidity-adjusted capital asset pricing model (LACAPM).

The model starts with the standard CAPM framework, in which the expected return of an asset (after transaction costs) is a function of its beta to the market return net of marketwide transaction costs:

$$\left(r_{t+1}^{i} - c_{t+1}^{i}\right) = r^{f} + \lambda_{i}\frac{cov_{t}(r_{t+1}^{i} - c_{t+1}^{i}, r_{t+1}^{M} - c_{t+1}^{M})}{var_{t}(r_{t+1}^{M} - c_{t+1}^{M})} \quad (4)$$

where r^f is the risk-free rate, r^i is the return to the i^{th} asset, c^i is the asset's liquidity cost, $r_{t+1}^{M} - c_{t+1}^{M}$ is the marketwide return net of trading costs and $\lambda_t = E_t(r_{t+1}^{M} - c_{t+1}^{M} - r^f)$ is the expected market risk premium. From the standard property of covariance,

$$cov(a + b, c + d) = cov(a, c) + cov(a, d) + cov(b, c) + cov(c, d).$$

Equation 4a yields the liquidity-adjusted CAPM:

$$E_t\left(r_{t+1}^{i}\right) = r^f + E_t\left(c_{t+1}^{i}\right) + \left(\beta_t^{CAPM} + \beta_t^{L1} - \beta_t^{L2} - \beta_t^{L3}\right)\lambda_t \quad (4a)$$

where β_t^{CAPM} is the standard CAPM beta and $(\beta_t^{CAPM} + \beta_t^{L1} - \beta_t^{L2} - \beta_t^{L3})$ represent different sensitivities of the asset's return and liquidity with respect to marketwide returns and liquidity. The betas are defined formally as

$$\beta^{CAPM} = \frac{cov(r^i, r^M)}{var(r^M - c^M)} \quad (4b)$$

$$\beta^{L1} = \frac{cov(c^i, c^M)}{var(r^M - c^M)} \quad (4c)$$

$$\beta^{L2} = \frac{cov(r^i, c^M)}{var(r^M - c^M)} \quad (4d)$$

$$\beta^{L3} = \frac{cov(c^i, r^M)}{var(r^M - c^M)} \quad (4e)$$

where we have dropped the time subscripts for notational simplicity. Equation 4b is the standard CAPM beta, Equation 4c is the beta of the asset's illiquidity with respect to marketwide illiquidity, Equation 4d is the beta of the asset's return with respect to marketwide liquidity, and Equation 4e is the beta of the asset's illiquidity with respect to the market return. Note that the signs are positive on 4b and 4c but negative on 4d and 4e. This is intuitive. Assets whose returns covary positively with the market return (4b) or whose illiquidity covaries positively with marketwide illiquidity (4b) should command higher returns as compensation for incurring adverse outcomes in negative states of the world. Conversely, assets whose returns covary positively with market illiquidity (4d) or whose illiquidity covaries positively with market returns should command a negative premium because such assets perform relatively well in the presence of negative wealth effects. *The key insight of Equation 4a is that investors will command a risk premium not just for the expected level of illiquidity,* $E_t\left(c_{t+1}^i\right)$*, as in Amihu and Mendelson, but also for the asset's covariance risk with respect to market return and market illiquidity.*

The authors estimate a liquidity risk premium of 1.1% between low and high expected liquidity stocks. Furthermore, they find that approximately 80% of the premium comes from β^{L3}, or the asset's illiquidity sensitivity to market returns, albeit estimated with considerable estimation error and model error. In other words, assets whose illiquidity is to the m increases when market returns are poor should command the greatest premium. Notably, the authors' illiquidity premium estimate of 110 basis points (bps) is calibrated using publicly traded equities. It's highly likely that the illiquidity premium is higher in the market for private assets, because secondary markets provide very limited opportunities to trade such assets.

Longstaff (2017)

Longstaff takes a radically different approach from the previously discussed models of the illiquidity premium. He starts by assuming that an asset is perfectly illiquid, meaning that it cannot be sold at any price over the investment horizon. An example of such a restriction might be a private fund vehicle that disallows the sale of fund shares over some lockup period. Models such as those of Amihud and Mendelson and Acharya and Pedersen make little sense in this context, as an infinite trading cost is incompatible with their formulations. Instead, the author takes the perspective that because the asset is unmarketable, the investor should be compensated for the forgone opportunity of selling it at a favorable intrahorizon valuation. Longstaff uses an options pricing framework to determine the opportunity cost associated with the inability to sell the asset at its expected high. This estimate should be considered an upper bound on the liquidity discount because disposing of the asset at the most favorable price over the horizon is the best-case scenario.

When an investor is precluded from selling an asset whose value is S, there is only a single stopping rule, which is to sell at the end of the investment period T, in which the investor receives S_T. However, if the investor is able to sell at the optimal stopping time $\tau < T$, he receives the value of the asset at that time, S_τ. The investor then invests his proceeds in a riskless cash asset and earns $S_\tau e^{r(T-\tau)}$ over the remainder of the investment period. Because the return to the investment is known with certainty after time τ, the value of the exchange option at time τ is a plain-vanilla put option with a strike price equal to $S_\tau e^{r(T-\tau)}$:

$$\max\left(0, S_\tau e^{r(T-\tau)} - S_T\right). \qquad (5)$$

Equation 5 says the value of being able to sell an otherwise unmarketable security at the optimal stopping time is equal to the value of a put option at the time of exercise. Note that the strike price is equal to the forward price of the asset at time τ. As such, the value of the exchange option today is simply a Black–Scholes option and the expected present value of Equation 5 is given by

$$E\left\{e^{-rT} S_\tau \left[N\left(\frac{\sqrt{\sigma^2(T-\tau)}}{2}\right) - N\left(-\frac{\sqrt{\sigma^2(T-\tau)}}{2}\right)\right]\right\} \qquad (6)$$

where the expectation is taken over the joint distribution of $S_\tau e$ and τ. Maximizing Equation 6 with respect to τ yields the closed-form expression for the upper-bound liquidity discount (relative to an otherwise liquid asset):

$$LD = S_0\left[N\left(\frac{\sqrt{\sigma^2 T}}{2}\right) - N\left(-\frac{\sqrt{\sigma^2 T}}{2}\right)\right]. \qquad (7)$$

The table below shows the illiquidity discounts from the Longstaff model for various lockup horizons and asset volatilities. For example, a 15% volatility asset with a three-year capital lockup has an illiquidity discount of 10.3%, which translates to a per-year discount of 3.4%. Recall that the model produces upper bounds for the illiquidity discount. In reality, given that investors would be unlikely to trade at the single most favorable valuation possible, the actual discount an investor would command should be less than the values in Exhibit 1.

Exhibit 1: Total and annual liquidity discount versus time horizon based on equation 7

	Total illiquidity discount					Per year illiquidity discount				
	Volatility					Volatility				
Horizon	5%	10%	15%	20%	25%	5%	10%	15%	20%	25%
1	2.0%	4.0%	6.0%	8.0%	9.9%	2.0%	4.0%	6.0%	8.0%	9.9%
2	2.8%	5.6%	8.4%	11.2%	14.0%	1.4%	2.8%	4.2%	5.6%	7.0%
3	3.5%	6.9%	10.3%	13.8%	17.1%	1.2%	2.3%	3.4%	4.6%	5.7%
4	4.0%	8.0%	11.9%	15.9%	19.7%	1.0%	2.0%	3.0%	4.0%	4.9%
5	4.5%	8.9%	13.3%	17.7%	22.0%	0.9%	1.8%	2.7%	3.5%	4.4%
6	4.9%	9.7%	14.6%	19.4%	24.1%	0.8%	1.6%	2.4%	3.2%	4.0%
7	5.3%	10.5%	15.7%	20.9%	25.9%	0.8%	1.5%	2.2%	3.0%	3.7%
8	5.6%	11.2%	16.8%	22.3%	27.6%	0.7%	1.4%	2.1%	2.8%	3.5%
9	6.0%	11.9%	17.8%	23.6%	29.2%	0.7%	1.3%	2.0%	2.6%	3.2%
10	6.3%	12.6%	18.7%	24.8%	30.7%	0.6%	1.3%	1.9%	2.5%	3.1%

Source: PIMCO as of date 30 November 2018

Ang, Papanikolaou and Westerfield (2014)

The liquidity problem can be complicated at will, such as in the mathematically cumbersome paper by Ang, Papanikolaou and Westerfield. The paper posits a risk-free asset as well as a risky liquid and illiquid asset. The fundamental difference between the liquid and illiquid assets is that the latter is subject to random, infrequent trading windows and cannot be pledged as collateral. The collateralization restriction prevents the investor from financing consumption by borrowing against the asset when it is in an illiquid state and is consistent with the fact that it can be difficult to obtain loans against assets for which there is low marketability. The investor chooses among the three assets to optimize the utility of lifetime consumption. Consumption can only be financed via the liquid asset portfolio, and because only liquid assets can be consumed, investors care about liquidity solvency ratios. The illiquidity premium depends on the average time between trading windows and is interpreted as the compensation an investor needs to trade the illiquid asset continuously.

The authors find, perhaps unsurprisingly, that the effect of illiquidity results in a lower allocation to the illiquid asset than in the fully liquid (Merton 1971) case. In the authors' baseline model, if the illiquid asset is assumed to be liquid, the optimal allocation is 60%. However, this falls rather dramatically to 37% for a one-year trading interval and to 5% for a 10-year trading interval. This fall in the allocation to the illiquid asset is largely mitigated, however, when the researchers remove the stochastic component of liquidity. Specifically, if the trading interval is known with certainty (not only known in expectation), the allocation to the illiquid asset falls only to 49% and 45% for one-year and 10-year trading intervals, respectively. Hence, it is the uncertainty associated with liquidity, rather than the expected time-horizon per se, that drives the allocation to the illiquid asset in their model.

The authors use their model to calibrate the illiquidity premium. To be clear, the illiquidity premium is the increase in expected return that investors should demand relative to an asset that is identical in every respect except that it is perfectly liquid. The table below shows the estimated illiquidity premium for various liquidity horizons, as well as the optimal allocation to the illiquid asset. For example, when the expected time between liquidity events is two years (for example, a two-year capital lockup), the investor allocates 24% of his portfolio to the illiquid asset and commands an illiquidity premium of 200 bps a year. Given that typical real estate, private equity and private loan investments have an average turnover of four to five years, an optimal allocation of 12% to illiquid assets sounds like a good rule of thumb. Of course, considering the number of parameters to calibrate and the degrees of freedom inherent in this class of models, the above numbers need to be met with healthy skepticism (see Exhibit 2).

Exhibit 2: Illiquidity premium and optimal allocation to the illiquid asset from Ang (2014)

Illiquidity premium as a function of liquidity frequency

Average time between liquidity events (years)	Illiquidity premium (basis points)
10	600
5	430
2	200
1	90
1/2	70

Optimal asset allocation to illiquid assets as a function of liquidity frequency

Average time between liquidity events (years)	Optimal asset allocation (%)
10	5
5	**11**
4	**13**
2	24
1	37
1/2	44
1/4	48
1/10	49
1/50	51
Continuous	59

Source: Ang, Andrew, Dimitris Papanikolaou, and Mark M. Westerfield. *Portfolio Choice with Illiquid Assets*. 2014.

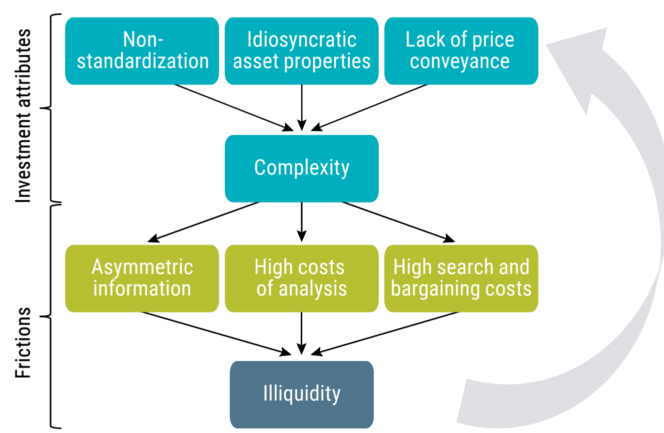
THE LINK BETWEEN COMPLEXITY AND ILLIQUIDITY

Complexity is a characteristic of many private investments, denoting the fact that assets that transact in private markets are often considerably more difficult to analyze and assess than those trading on public exchanges. Complexity stems from certain attributes that can be unique to the private asset market: nonstandardization, idiosyncratic characteristics of the underlying investments and the inability to rely on past prices because of infrequent transactions. Physical real estate investments, for example, trade infrequently and are often highly idiosyncratic, necessitating assessment of not just the prospective cash flows of the property but also expectations for the area's economic growth and the evaluation of local rules and regulations.

The costs associated with complexity are not borne only by potential investors. Because the buyer base for complex assets is limited, the search costs associated with locating buyers can be onerous for sellers. This effort, which usually entails working with an intermediary such as an investment bank or broker, involves considerable time and resources, and imposes on sellers not just search costs but also the opportunity costs of forgone investment opportunities. High search costs generally favor the prospective buyer, particularly if the buyer base is limited. In this context, Duffie, Gârleanu and Pedersen (2007) developed a model whereby investors are differentiated by their preference for asset ownership and derive the impact on asset prices when the identification of, and negotiation with, buyers is onerous. The authors find that, under natural conditions, prices are lower if buyers and sellers have difficulty finding each other, sellers have less bargaining power or the fraction of qualified owners is lower.

These supply and demand frictions – high analysis and search costs – can cause significant delays between transactions in private markets, leading directly to illiquidity. Illiquidity means that there may be little or no informational price conveyance from the observation of prior transactions; past trades may be weeks or months old, if they exist at all. The result is a critical feedback effect: Long delays between transactions mean that prices are often stale and thus only quasi-informational. This, in turn, creates complexity for future transactions, as tomorrow's buyers cannot rely on past trades in their assessment of fair value. This feedback effect is illustrated in Exhibit 3.

Exhibit 3: Private market transaction illiquidity cycle



Source: PIMCO

THE IMPORTANCE OF SCALE

In the pursuit of complex opportunities, investors incur substantial financial costs. Often, teams of analysts, accountants and lawyers are required to evaluate the prospects for the underlying investments, and in the case of structured products, such as collateralized debt obligations, to assess the technicalities of the instruments. Such costs, however, are largely fixed in nature, meaning that they typically do not increase one-to-one with the size of the transaction; it takes roughly the same amount of resources to assess the prospect for a $1 million investment as it does for a $100 million allocation. As the investment size increases, research costs remain roughly fixed, while the dollar returns increase linearly. Thus, a threshold exists whereby, once a manager meets a sufficient level of investment scale, complex private investments can be profitable in expectation. Managers whose scale falls below this threshold, however, will be unable to recoup the fixed costs of analysis through future investment returns. This gives larger asset managers a distinct advantage over their peers with fewer resources.

In this context, Grossman and Stiglitz (1980) developed a model in which investors can become informed about the prospects for a risky asset by paying a fixed cost, allowing them to observe private information. Paying such a cost allows the investor to receive a more precise signal, both in terms of the asset's expected return and its variance. This is equivalent to an investor committing significant resources to analyzing a potential deal, thereby removing uncertainty and better identifying an investment's prospects. The authors show that the higher the cost associated with becoming informed, the smaller the equilibrium percentage of prospective buyers who choose to pay it. If the acquisition costs of information are high – such as in the market for complex private investments – the pool of buyers is naturally limited to those with the scale to justify those costs.

In addition to the scale advantage larger investors possess, there is a second-order benefit, which can be equally important. When pricing complex assets, prospective buyers will apply an additional risk premium to account for the fact that the information between the buyer and the seller is asymmetric (in favor of the seller), even after a detailed examination of the investment's prospects. However, investors with greater resources and a deep bench of investment acumen can reduce the unbalanced information relative to buyers with fewer resources, allowing the former to reduce the required risk premium associated with informational asymmetry. This, in turn, may allow better-informed buyers to outbid competitors when deals are deemed attractive, potentially resulting in superior risk-adjusted returns for the end investor.

To formalize this notion, consider a risk-neutral investor who is considering paying the fixed cost C_0 to acquire information concerning the value of a potential investment. If the investor chooses to pay C_0, he receives a signal about the true value of the asset. Hence, the investor chooses to pay or not to pay, and earns the following expected payoffs for each decision:

$$E[\omega_t] = \omega_{t-1}(1+\mu) \tag{5a}$$

$$E[\omega_t] = \omega_{t-1}(1+\mu+\lambda) - C_0 \tag{5b}$$

where ω is the investor's wealth, μ is the unconditional expected return that all investors can observe and λ is the additional return the investor expects to earn from paying C_0. If the investor doesn't pay the research cost C_0, then he expects to earn the unconditional return μ, as in Equation 5a. If he does pay, then his expected return increases by λ, but his wealth decreases by the fixed research costs C_0, as shown in Equation 5b. It is straightforward to show that the investor chooses to pay the research cost if

$$\omega_{t-1} > \frac{C_0}{\lambda}. \tag{6}$$

Equation 6 clearly conveys the importance of scale. It says that investors are willing to undertake research costs (and hence earn a higher expected return) if their wealth is sufficiently high, the costs of research are sufficiently low or the expected payoff from the research is high. All investors face roughly the same cost-to-payoff ratio C_0/λ, so the main determinant of whether investors are willing to incur research costs is if they have a sufficiently large investment base over which to amortize those costs. Hence, only investors with sufficient asset size can justify high research costs and ultimately earn the complexity premium.

CONCLUSION

Illiquidity and complexity are two fundamental characteristics of private market investments that differentiate them from assets that trade publicly. These factors limit the set of potential buyers to those with the patience to amortize transaction costs over a sufficiently long holding horizon, and to those with sufficient scale to justify the high research costs associated with deciphering complex investments. Importantly, illiquidity and complexity are not independent but, rather, related by the fact that complex investments often trade at lower frequencies. This means that the price discovery process cannot reliably look to past prices as an indication of fair value. Hence, a feedback mechanism exists in which complexity leads to illiquidity, which, in turn, creates complexity. Ultimately, the set of investors who possess both the necessary patience and the scale to undertake private market investments is limited.

Fortunately, this subset of investors has the ability to extract both the illiquidity premium and the complexity premium in the market for private investments.

TECHNICAL APPENDIX

Derivation of Equation 2 (Amihud and Mendelson, 1986)

Investors are assumed to be risk neutral. Consider a set of J investor types $j = 1, \ldots, J$, each with a liquidity shock probability μ_j and I security types $i = 1, \ldots, I$, each with varying degrees of transaction costs. Costs are ordered such that Asset 1 has the lowest costs and I has the highest costs. In other words, $C^1/d^1 \leq C^2/d^2 \leq \cdots \leq C^I/d^I$, where c^i and d^i are the costs and dividend of the i^{th} asset. Each investor j chooses his holding in each asset by maximizing the after-cost return of $(d^i - \mu^j C^i)/P^i$. Note that each security's return is a function of properties of the asset as well as the investor.

The authors show that, in a competitive equilibrium, investors specialize according to their trading intensity. The exact allocations in the economy, as well as prices (and thus returns), will depend on each investor's initial endowment. The investors with the highest liquidity needs hold the risk-free assets and the least costly illiquid assets. As such, these investors earn the risk-free rate and thus $r^f = (d^i - \mu^1 C^i)/P^i$, where μ^1 represents the trading intensity up to the last illiquid asset held by Investor Type 1. This naturally implies that $P^i = (d^i - \mu^1 C^i)/r^f$ so that prices for the most liquid illiquid assets are set by investors with the highest trading intensity. The same logic holds for the next set of investors, with a higher liquidity tolerance. If we denote the return earned by the next cohort as r^2, then they earn a return equal to $(d^i - \mu^2 C^i)/P^i$. However, this cohort earns a risk premium above the risk-free rate; this return can be expressed as $r^f(d^i - \mu^2 C^i)/(d^i - \mu^1 C^i) > r^f$. This process continues until all endowments are expended on the investment asset, with each successive investment type earning a higher risk premium. This yields Equation 2 in the paper.

Derivation of Equation 4 (Acharya and Pedersen, 2005)

Consider the following first-order autoregressive processes for the dividend D and illiquidity cost C: $D_t = \overline{D} + \rho^D(D_{t-1} - \overline{D}) + \varepsilon_t$ and $C_t = C + \rho^C(C_{t-1} - C) + \eta_t$. Each investor n maximizes the following utility function: $max_{y^n}^n E_t(W_{t+1}^n) - 0.5A^n var_t(W_{t+1}^n)$, where $W_{t+1}^n = (P_{t+1}+D_{t+1} - C_{t+1})'y^n + r^f(e_t^n - P'_t y^n)$, y^n is the vector of the investor's allocation to the risky assets, A^n is the investor's risk-aversion coefficient and e_t^n is her endowment. The equilibrium condition then becomes $P_t = 1/r^f[E_t(P_{t+1}+D_{t+1} - C_{t+1}) - Avar_t\ P_{t+1}+D_{t+1} - C_{t+1})S$, where S is the vector of the number of shares and $A = \sum_n(1/A^n)^{-1}$. Substituting in the stochastic process for dividends and costs yields the unique stationary linear equilibrium $P_t = \gamma + \frac{\rho^D}{r^f - \rho^D}D_t - \frac{\rho^C}{r^f-\rho^C}C_t$, where y is a complex function of the relevant model parameters. With this price, conditional expected net returns are normally distributed and any investor n holds a fraction A/A^n of the market portfolio, so there is no short selling. Because investors have mean-variance preferences, the conditional CAPM holds for net asset returns.

Derivation of Equation 7 (Longstaff, 2017)

Equation 7 yields an upper bound, as it represents the worst-case illiquidity scenario; the author assumes that the asset is perfectly illiquid and that the investor has the foresight to be able to sell the asset at its intrahorizon high if it were perfectly liquid. Equation 7 is obtained by solving for the investor's optimal stopping rule τ^*. Hence, the investor solves the following maximization problem:

$$max_\tau E_0\left[e^{-r\tau}S_\tau\left[N\left(\frac{\sqrt{\sigma^2(T-\tau)}}{2}\right) - N\left(-\frac{\sqrt{\sigma^2(T-\tau)}}{2}\right)\right]\right].$$

We can prove that this expression is maximized when τ takes its lowest value τ = 0. Considering the optimization problem as

$$\max_\tau \mathbb{E}[e^{-r\tau}S_\tau f(\tau)]$$

where $f(\tau) = N\left(\frac{\sqrt{\sigma^2(T-\tau)}}{2}\right) - N\left(-\frac{\sqrt{\sigma^2(T-\tau)}}{2}\right)$

we already can note that by construction,

$$f(\tau) \leq f(0)$$
$$S_0 f(0) \leq \max_\tau e^{-r\tau}S_\tau f(\tau).$$

Therefore, as S_τ is a martingale and by the optional stopping theorem, we also have

$$\begin{aligned}\mathbb{E}[e^{-r\tau}S_\tau f(\tau)] &\leq f(0)\mathbb{E}[e^{-r\tau}S_\tau].\\ &\leq f(0)S_0\end{aligned}$$

Hence, the maximization program is achieved when τ = 0 and we have an upper bound equal to

$$LD = S_0\left[N\left(\frac{\sqrt{\sigma^2 T}}{2}\right) - N\left(-\frac{\sqrt{\sigma^2 T}}{2}\right)\right].$$

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